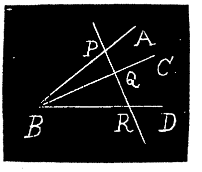
III. Solution by the PROPOSER.

Let ABC and CBD be the two angles x and y. Draw any line perpendicular to BC meeting AB, BC, BD at P, Q, R.



Then $\triangle BPR = \triangle BPQ + \triangle BQR$.

$2\triangle BPR = BP \times BR \times \sin PBR = BP.BR.\sin(x+y)$.

$2\triangle BPR = BP \times BQ \times \sin PBQ = BP.BQ.\sin x$.

$2\triangle BQR = BQ.RB.\sin y$.

Hence $BP.BR\sin(x+y) = BP.BQ\sin x + BQ.BR\sin y$.

Dividing by $BP.BR$, $\sin(x+y) = \dfrac{BQ}{BR}\sin x + \dfrac{BQ}{BP}\sin y = \cos y\sin x + \cos x\sin y$.

CALCULUS.

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84. Proposed by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Find the equation of the curve upon which a given ellipse must roll in order that one of its foci may describe a straight line.

Solution by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Let θ be the angle between the major axis of the ellipse and the radius vector from the focus to the point of contact; r the length of this radius, a, b, semi-axes, e eccentricity. Then

$$r = \frac{a(1-e^2)}{1-e\cos\theta}.$$

Let the axis of x be taken parallel to the given line. Then since the point of contact is the instantaneous center, the radius vector will always be perpendicular to the axis of x, and hence $r+y=a(1-e)$, y being the ordinate of a point on the required curve. We have also

$$r\frac{d\theta}{dr} = -\frac{dx}{dy}.$$

Eliminating r and θ, we find

$$y - \sqrt{b^2\left(\frac{dy}{dx}\right)^2 + a^2} = ae.$$

Whence $\dfrac{ae-y}{b} = \sin\left(\dfrac{x}{b} + \sin^{-1}\dfrac{ae}{b}\right)$, or $y' = b\sin\left(\dfrac{x'}{b}\right)$.

86. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.

Prove that the curve whose normal equals its radius of curvature drawn in an opposite direction, is the catenary $y = c\cosh(x/c)$.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

We should have

$$\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=y\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}, \text{ whence} \frac{2\frac{dy}{dx}\frac{dy^2}{dx^2}}{1+\frac{dy^2}{dx^2}}=\frac{2}{y}\frac{dy}{dx}.$$

Multiplying by dx and integrating and correcting,

$$1+\frac{dy^2}{dx^2}=\frac{y^2}{c^2}.$$

This gives $dx=\dfrac{cdy}{\sqrt{y^2-c^2}}$; whence integrating and correcting again,

$$x=c\log\left(\frac{y+\sqrt{(y^2-c^2)}}{c}\right), \text{ or } y=c\cosh(x/c).$$

II. Solution by MARY M. BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

Our differential equation, formed by substituting the values of normal and ρ, the radius of curvature is

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \ldots\ldots(1).$$

$$\therefore y\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right] \ldots\ldots(2).$$

Now let $p=\dfrac{dy}{dx}$, then $\dfrac{d^2y}{dx^2}=\dfrac{dp}{dx}=\dfrac{dp}{dy}.\dfrac{dy}{dx}=\dfrac{dp}{dy}.p.$

Substituting, $y\,p\,\dfrac{dp}{dy}=(1+p^2)\ldots\ldots(3).$

$\therefore \dfrac{pdp}{1+p^2}=\dfrac{dy}{y}.$ Integrating,

$\log y=\frac{1}{2}\log(p^2+1)+\log c_1$, or $\log(y/c_1)=\log\sqrt{(p^2+1)}\ldots\ldots(4).$
$\therefore y/c_1=\sqrt{(p^2+1)}\ldots\ldots(5).$
Squaring and clearing, $c_1p^2=y^2-c_1^2\ldots\ldots(6).$

$$p^2=\frac{y^2-c_1^2}{c_1^2}, \text{ or } p=\frac{\sqrt{(y^2-c_1^2)}}{c_1^2}\ldots\ldots(7).$$

Then, substituting for p, and separating,

$$\frac{dy}{\sqrt{y^2-c_1^2}}=\frac{dx}{c_1^2}\ldots\ldots(8). \text{ Integrating, } \log[y+\sqrt{(y^2-c_1^2)}]=\frac{x}{c_1^2}+\log c_2 \ldots.(9).$$

$$\therefore \frac{y+\sqrt{(y^2-c_1^2)}}{c_2}=e^{x/c_1}\ldots\ldots(10).$$

Transposing and squaring, $y^2-c_1^2=c_2^2e^{2x/c_1}-c_2e^{x/c_1}y+y^2\ldots\ldots(11)$.

$$\therefore y=\frac{c_2^2e^{2x/c_1}+c_1^2}{c_22e^{x/c_1}}\ldots.\,.(12), \text{ or } y=\frac{c_2^2e^{x/c_1}+c_1^2e^{-(x/c_1)}}{2c_2}\ldots\ldots(13).$$

Let $c_2=c_1$, thus moving the origin to the right. Then $y=\dfrac{c_1[e^{x/c_1}+e^{-(x/c_1)}]}{2}$ $\ldots.(14)$, or $y=c_1\cosh(x/c_1)$, which is the equation of the catenary.

Also solved by *J. W. YOUNG*.

87. Proposed by MARY M. BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

Integrate $(px-y)(py+x)=h^2p$, where $p=dy/dx$.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

We have $(px-y)(py+x)=h^2p\ldots\ldots(1)$, or

$$p^2xy-py^2+px^2-xy=h^2p\ldots\ldots(2).$$

Multiplying by y and arranging,

$$y^2=\left(y\frac{dy}{dx}\right)^2-\left(\frac{y^3}{x^2}-xy+\frac{h^2y}{x}\right)\frac{dy}{dx}\ldots\ldots(3).$$

Putting $y^2=y'$, $x^2=x'$,

$$y'=x'\left(\frac{dy'}{dx'}\right)^2-y'\frac{dy'}{dx'}+x'\frac{dy'}{dx'}-\frac{dy'}{dx'}h^2\ldots\ldots(4),$$

or $y'=p'x'-\dfrac{h^2p'}{p'+1}\ldots\ldots(5)$, Clairaut's Form, giving $y^2-cx^2=-\dfrac{ch^2}{c+1}\ldots.(6)$.

II. Solution by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Divide by p, differentiate and reduce to

$$xy\frac{dp}{dx}+(px-y)p=0.$$